Signature retake exam – mock exam

Name: ....

Data structures and algorithms (GEMAK117-MA)

May 21, 2024

Neptun code: ....

Part 1: theoretical questions (15 points)

I will ask a few definitions and theorems and one algorithm from the glossary.

Exercise 1 (6 points)**.** State the following definitions (1 point each):

a) whole quotient, div operation

$a \text{ div } b = \left\lfloor \frac{a}{b} \right\rfloor$

b) small o notation

f and g are functions on $\mathbb{N}$. $f(n) = o(g(n))$ if $\dfrac{f(n)}{g(n)} \to 0$ as $n \to \infty$.

c) algorithm

a step-by-step calculation to solve a problem

d) Fibonacci numbers

a number sequence defined recursively: $F_0 = 0$, $F_1 = 1$, then $F_n = F_{n-1} + F_{n-2}$.

e) congruence

$a \equiv b \mod n$, if a and b have the same remainder when divided by n.

f) multiplicative inverse

$a^{-1} \mod n$ only exists when $\gcd(a, n) = 1$. Then it's the single solution $x \in [0, n)$ of the linear congruence equation $ax \equiv 1 \mod n$.

Exercise 2 (6 points)**.** State the following theorems (2 points each):

a) reduction theorem (of the greatest common divisor)

a and b are whole numbers. $\gcd(a, b) = \gcd(a - b, b)$.

b) number of digits (in base b)

x positive, whole number has $\lfloor \log_b(x) \rfloor + 1$ digits in base b.

c) the "master theorem"

given a recursive equation $T(n) = aT\left(\dfrac{n}{b}\right) + f(n)$.

suppose $a \geq 1, b > 1$, f is a function $\mathbb{N} \to \mathbb{R}^+$.

define $p = \log_b(a)$, $g(n) = n^p$ test polynomial.

a) if $f(n)$ grows polynomially slower than $g(n)$, then $T(n) = \Theta\big(g(n)\big)$.

b) if $f(n) = \Theta(g(n))$, then $T(n) = \Theta\big(g(n) \cdot \log(n)\big)$.

c) if $f(n)$ grows polynomially faster than $g(n)$, and also $f(n)$ satisfies regularity:

for some $c < 1$, $c \cdot f(n) \leq af\left(\frac{n}{b}\right)$,

then $T(n) = \Theta\big(f(n)\big)$.

Exercise 3 (3 points)**.** Write down the algorithm for **modular exponentiation**.

MOD_EXP(a,b,n,@c)
// INPUT: whole numbers a,b,n
// OUTPUT: $c = (a^b \mod n)$
write the exponent b in base 2: $b_n b_{n-1} \dots b_1 b_{0(2)}$
$c \leftarrow 1$
FOR k ← n DOWNTO 0 DO // that is: read binary digits of b left to right
    $c \leftarrow c^2 \mod n$
    IF $b_k = 1$ THEN
        $c \leftarrow c \cdot a \mod n$
RETURN(c)

PART 2: EXERCISES (15 POINTS)

I will pick 3 of the 6 exercise types seen in the practical midterm.

Exercise 4 (5 points)**.** Using the extended Euclidean algorithm, calculate the greatest common divisor $d^\star$ of $a = 410$ and $b = 305$, then write $d^\star$ as a linear combination (with whole number coefficients) of a and b.

i	a	b	q	r	x	y
0	410	305	1	105	-29	$10 - (-29) \cdot 1 = 39$
1	305	105	2	95	10	$-9 - 10 \cdot 2 = -29$
2	105	95	1	10	-9	$1 - (-9) \cdot 1 = 10$
3	95	10	9	5	1	$0 - 1 \cdot 9 = -9$
4	10	5	2	0	0	$1 - 0 \cdot 2 = 1$
5	5	0			1	0

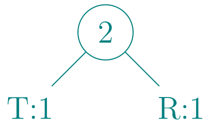
$d = \gcd = 5$, coefficients: $x = -29$, $y = 39$.
double checking: $x \cdot a + y \cdot b = -29 \cdot 410 + 39 \cdot 305 = 5 = d$, ok.

Exercise 5 (5 points)**.** Encode the message `PETER PAN` using the Huffman encoding. What is the coded message, and what is the average code length per character?

stage 1: frequencies
P:2, E:2, T:1, R:1, A:1, N:1

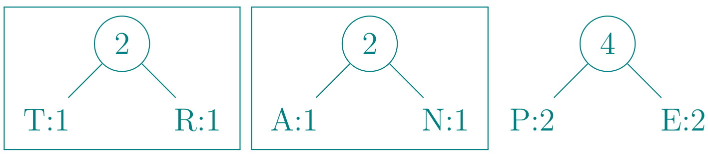
stage 2: build code tree – keep objects in increasing order, always combine two smallest
0) T:1 , R:1 , A:1, N:1, P:2, E:2

1) A:1 , N:1 , P:2, E:2

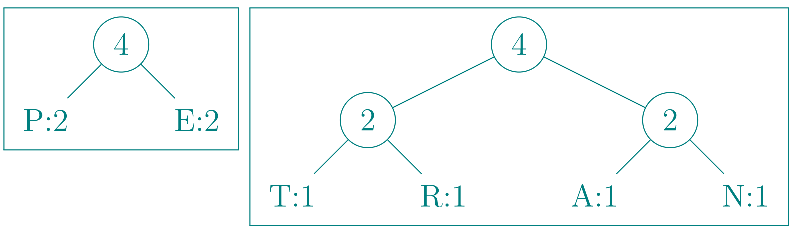


2) P:2 , E:2 ,

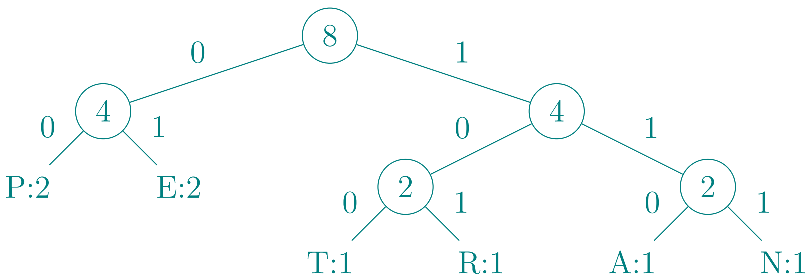
3)



4)



5)



also wrote 0-1 bits on edges in above complete code tree.

stage 3: codes of each letter
P: 00
E: 01
T: 100
R: 101
A: 110
N: 111

stage 4: encoded message
`P  E  T   E  R  P  A  N`
`00 01 100 01 101 00 110 111` (total code length: 20 bits)
average code length: 20 bits / 8 characters = 2.5.

Exercise 6 (5 points)**.** Sort the array A = $[1, 4, 5, 3, 4, 1, 5, 1]$ using `BINSORT` (aka counting sort).

stage 1: frequencies

C = [3,0,1,2,2] – three 1s, no 2s, one 3, two 4s, two 5s.

stage 2: cumulative frequencies

$\tilde{C}$ = [3,3,4,6,8] – calculate as number to the left + corresponding number in C.

stage 3: construct new array B – read A from right to left, and according to value, use $\tilde{C}$ as index guide

0) B = [. , . , . , . , . , . , . , .]

1) j=8, value $A_8 = 1$. index: $\tilde{C}_1 = 3$, place value 1 at index 3 in B, then decrease $\tilde{C}_1$:

B = [. , . , 1 , . , . , . , . , .], $\tilde{C}$ = [$\boxed{2}$,3,4,6,8]

2) j=7, value $A_7 = 5$. index: $\tilde{C}_5 = 8$, place value 5 at index 8 in B, then decrease $\tilde{C}_5$:

B = [. , . , 1 , . , . , . , . , 5], $\tilde{C}$ = [2,3,4,6,$\boxed{7}$]

3) j=6, value $A_6 = 1$. index: $\tilde{C}_1 = 2$ (previously decreased!), place value 1 at index 2 in B, then decrease $\tilde{C}_1$:

B = [. , 1 , 1 , . , . , . , . , 5], $\tilde{C}$ = [$\boxed{1}$,3,4,6,7]

and so on...

eventually: B = [1,1,1,3,4,4,5,5]

evolution of $\tilde{C}$:

```
C = [3,3,4,6,8]
     2   3 5 7
     1     4 6
     0
```

Scoring

Total 30 points, pass: 15+ points.